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UNIQUE ERGODICITY OF FLOWS ON HOMOGENEOUS SPACES

BY

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ABSTRACT

Let G be a unimodular Lie group, Γ a co-compact discrete subgroup of G and 'a' a semisimple element of G. Let T_a be the map $g\Gamma \to ag\Gamma : G/\Gamma \to G/\Gamma$. The following statements are pairwise equivalent: (1) $(T_a, G/\Gamma, \omega)$ is weak-mixing. (2) $(T_a, G/\Gamma)$ is topologically weak-mixing. (3) $(G^u, G/\Gamma)$ is uniquely ergodic. (4) $(G^u, G/\Gamma, \omega)$ is ergodic. (5) $(G^u, G/\Gamma)$ is point transitive. (6) $(G^u, G/\Gamma)$ is minimal. If in addition G is semisimple with finite center and no compact factors, then the statement "$(T_a, G/\Gamma, \omega)$ is ergodic" may be added to the above list.

0. Introduction

In this paper we give a different proof of and extend slightly the main result obtained by Bowen in [2]. The latter states: Let G be a unimodular Lie group, Γ a cocompact discrete subgroup of G and a a semisimple element of G such that the flow $(T_a, G/\Gamma, \omega)$ is weak-mixing. Then $(G^u, G/\Gamma)$ is uniquely ergodic. (See section 2 for definitions.) We prove (2.4) that in the above situation the following statements are pairwise equivalent: (1) $(T_a, G/\Gamma, \omega)$ is weak-mixing. (2) $(T_a, G/\Gamma)$ is topologically weak-mixing. (3) $(G^u, G/\Gamma)$ is uniquely ergodic. (4) $(G^u, G/\Gamma, \omega)$ is ergodic. (5) $(G^u, G/\Gamma)$ is point transitive. (6) $(G^u, G/\Gamma)$ is minimal.

If in addition G is semisimple with finite center and no compact factors, then the statement "$(T_a, G/\Gamma, \omega)$ is ergodic" may be added to the above list. (See Proposition 3.4).

Our proof resulted from an attempt to apply the method used by Marcus in [4] to the situation considered by Bowen. In [4] Marcus proves that the horocycle flow (ϕ_s) is uniquely ergodic by showing that $\lim_{T\to\infty}(1/T)\int_0^T h\circ\phi_s(x)ds$ is constant for every continuous h. He does this by studying

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$\lim_{n\to\infty}(1/2^n)\int_0^{2^n} h\circ\phi_s\circ f_{t_n}(x)ds = \lim_{n\to\infty}\int_0^1 h\circ f_{t_n}\circ\phi_s(x)ds$ and exploiting the way in which the geodesic flow (f_t) and the horocycle flow (ϕ_s) are related. In our discussion G^u plays the role of the horocycle flow and T_a the role of the geodesic flow.

The basic ideas involved in our approach are abstracted and discussed in section 1. There we consider a flow (X, T) with compact phase space X and locally compact phase group T together with a homeomorphism ϕ of X such that $\phi^{-1}T\phi = T$ and a supported Borel measure ω invariant with respect to both T and ϕ. The essential "ingredient" is the existence of a compact neighborhood V of the identity of T such that (i) the maps $t\to\phi^n t\phi^{-n}: V\to T$ $(n\geqq 0)$ converge uniformly to the constant map $t\to e: V\to T$ and (ii) the family of maps $x\to(1/\mu(V))\int_V f(xt\phi^n)d\mu(t): X\to\mathbf{C}$ $(n\geqq 0)$ is equicontinuous $(f\in C(X))$. (Here μ is Haar measure on T.)

Under these assumptions an abstract version of the aforementioned theorem is obtained (see 1.10). This is applied in sections 2 and 3 to get the results discussed above.

1. General theory

Here we discuss an abstract version of the results obtained by Bowen in [2].

1.1 Notation. The following notation will be used throughout this section.

(a) (X, T) is a flow with compact T_2 phase space X and locally compact phase group T with Haar measure μ.

(b) ϕ is a homeomorphism of X such that $\phi t\phi^{-1}\in T$ $(t\in T)$. (Here T is viewed as a subgroup of the group of homeomorphisms of X.)

(c) ω will denote a Borel probability measure on X invariant under ϕ and T such that $\omega(N)>0$ for all open subsets N of X.

(d) $\mathscr{H}$ will denote the Hilbert space $L_2(X,\omega)$ and U, U_t $(t\in T)$ will denote the unitary operators induced by ϕ and t $(t\in T)$ respectively.

(e) The linear map $L: C(X)\to C(X)$ is defined by $(Lf)(x) = (1/\mu(V))\int_V f(xt)d\mu(t)$, where V is a fixed compact neighborhood of the identity of T. It is easy to see that L has a continuous linear extension (also denoted L) to $\mathscr{H}$ when $C(X)$ is viewed as a subset thereof.

1.2 Basic Assumptions. 1. The sequence of maps $t\to\phi^n t\phi^{-n}: V\to T$ $(n\geqq 0)$ converges uniformly to the constant map $t\to e: V\to T$.

2. The family $(LU^n f\mid n\geqq 0)$ is equicontinuous for all $f\in C(X)$. (Note $LU^n f$ is just the map $x\to(1/\mu(V))\int_V f(xt\phi^n)d\mu(t): X\to\mathbf{C}$.)

1.3 LEMMA. *For each $f \in C(X)$ let there exist a sequence of positive integers $J(f)$ and a constant $c(f)$ such that the sequence $(U^{-n}LU^nf \mid n \in J(f))$ converges pointwise to $c(f)$. Then the flow (X, T) is uniquely ergodic.*

PROOF. Let m be a T-invariant Borel probability measure on X and let $f \in C(X)$.

By Fubini's theorem

$$\frac{1}{\mu(V)}\int_V \int f(x\phi^{-n}t\phi^n)dm(x)d\mu(t) = \frac{1}{\mu(V)}\int\int_V f(x\phi^{-n}t\phi^n)d\mu(t)dm(x)$$

for all n. Since m is T-invariant and $\phi^{-n}t\phi^n \in T$ $(t \in T)$, the left hand side of the above equation is just $m(f)$. Thus $m(f) = \int U^{-n}LU^nf(x)dm(x)$ which converges to $m(c(f)) = c(f)$. The proof is completed.

1.4 COROLLARY. *For each $f \in C(X)$ let there exist a sequence of positive integers $J(f)$ and a constant $c(f)$ such that the sequence $(LU^nf \mid n \in J(f))$ converges uniformly to $c(f)$. Then the flow (X, T) is uniquely ergodic.*

PROOF. This follows from 1.3 and the fact that $(U^{-n}LU^nf)(x) = (LU^nf)(x\phi^{-n})$ $(x \in X, n \in \mathbf{Z})$.

1.5 LEMMA. *The sequence of operators $(U^nLU^{-n}n \geqq 0)$ converges strongly to the identity.*

PROOF. Let $f \in \mathcal{H}$ and $\varepsilon > 0$. Then by 1 of 1.2 there exists a positive integer N such that $\|U^nU_tU^{-n}f - f\| < \varepsilon$ $(t \in V, n \geqq N)$. (Recall that the map $s \to U_sf \colon T \to \mathcal{H}$ is continuous.) Then

$$\begin{aligned}\|U^nLU^{-n}f - f\| &= \left\| \frac{1}{\mu(V)}\int_V U^nU_tU^{-n}fd\mu(t) - f\right\| \\ &= \frac{1}{\mu(V)}\left\|\int_V (U^nU_tU^{-n}f - f)d\mu(t)\right\| \\ &\leqq \frac{1}{\mu(V)}\int_V \|U^nU_tU^{-n}f - f\|\, d\mu(t) \\ &< \varepsilon.\end{aligned}$$

1.6 COROLLARY. *Let $Uf = \lambda f$. Then $Lf = f$.*

PROOF. Let $\varepsilon > 0$. By 1.5 there exists n with $\|LU^{-n}f - U^{-n}f\| = \|U^nLU^{-n}f - f\| < \varepsilon$. But $\|LU^{-n}f - U^{-n}f\| = \|\lambda^{-n}Lf - \lambda^{-n}f\| = \|Lf - f\|$.

1.7 Remarks. 1. Corollary 1.6 says that L is the identity on the set of L_2 ϕ-eigenfunctions. Thus if $\mathcal{H}_P$ denotes the closed subspace spanned by the latter, $Lf = f$ $(f \in \mathcal{H}_P)$. Another way of saying this is that $LP = P$ where P is the projection onto $\mathcal{H}_P$. The elements of $\mathcal{H}_P$ are called *L_2-almost periodic functions* (with respect to ϕ). For a discussion see [3].

1.8 Lemma. *Let $f \in C(X)$ with $Pf = f$. Then f is an almost periodic function, i.e. $(U^n f \mid n \in \mathbf{Z})$ is a relatively compact subset of $C(X)$.*

Proof. It is well known that it suffices to show that $(U^n f \mid n \geqq 0)$ is equicontinuous.

By 2 of 1.2 $(LU^n f \mid n \geqq 0)$ is equicontinuous. Since $Pf = f$, $PU^n f = U^n f$ for all n (see [3]). Hence $LU^n f = U^n f$ (a.e. ω) by 1.7, whence $LU^n f = U^n f$ since $LU^n f$ and $U^n f$ are continuous and $\omega(N) > 0$ for all nonempty open sets N.

1.9 Lemma. *Let the flow (X, ϕ) be topologically weak mixing. Then (X, T) is uniquely ergodic and (X, ϕ, ω) is weak mixing.* (Note: The hypothesis states that there are no continuous ϕ-eigenfunctions and the second part of the conclusion that there are no L_2 ϕ-eigenfunctions.)

Proof. Let $f \in C(X)$. Then there exists a sequence of positive integers (n_i) with $(U^{n_i} f)$ converging weakly to Pf [3]. By 2 of 1.2 we may assume that $(LU^{n_i} f)$ converges uniformly to the continuous function g.

Since $(LU^{n_i} f) \to LPf = Pf$ weakly, $Pf = g$ whence $Pg = P^2 f = Pf = g$. By 1.8 g is an almost periodic function and so must be constant since there are no nonconstant continuous eigenfunctions. The first part of the conclusion now follows from 1.4.

The above proof shows that $P(C(X)) \subset \mathbf{C}$. Hence $P(\mathcal{H}) \subset \mathbf{C}$ and (X, ϕ, ω) is weakly mixing.

1.10 Proposition. *Statements* 1 *through* 4 *are equivalent, they are all implied by* 5, *and when X is metrizable or T is amenable all five statements are equivalent. In the latter case* 6 *may be added to the list of equivalences.* 1. *(X, ϕ, ω) is weak-mixing.* 2. *(X, ϕ) is topologically weak-mixing.* 3. *(X, T, ω) is uniquely ergodic.* 4. *(X, T, ω) is ergodic.* 5. *(X, T) is point transitive (i.e., there exists $x_0 \in X$ with* $\operatorname{cls}(x_0 T) = X$). 6. *$(X, T)$ is minimal.*

Proof. That 1 implies 2 follows immediately from the definition of weak-mixing and topologically weak-mixing.

Lemma 1.9 shows that 2 implies 1 and 3.

Clearly 3 implies 4.

Assume 4 and let f be a continuous ϕ-eigenfunction with eigenvalue λ. Let (n_i) be a sequence of positive integers tending to infinity with $\lim \lambda^{-n_i} = 1$. Then

$$f(xt) = \lim \lambda^{-n_i} f(xt) = \lim f(xt\phi^{-n_i}) = \lim f(x\phi^{-n_i}\phi^{n_i}t\phi^{-n_i})$$
$$= \lim f(x\phi^{-n_i}) = \lim \lambda^{-n_i} f(x) = f(x)$$

by 1 of 1.2. Thus $f(xt) = f(x)$ $(x \in X,\ t \in T)$. Since (X, T, ω) is ergodic and $\omega(N) > 0$ for all nonvacuous open sets, f is constant. Thus 4 implies 2. When X is metrizable 4 together with the fact that ω is supported implies 5.

Assume 5 and again let f be a continuous ϕ-eigenfunction. Then as above $f(xt) = f(x)$ $(x \in X, t \in T)$ whence $f(x) = f(x_0)$ $(x \in X)$ and so 5 implies 2.

It is well known that when T is amenable statement 3 together with the fact that $\omega(N) > 0$ for all nonempty open sets N implies that (X, T) is minimal.

The proof is completed.

2. Homogeneous spaces

In this section we apply the results of section 1 to the action of certain subgroups of a connected Lie group G on the homogeneous space G/Γ where Γ is a cocompact discrete subgroup of G. (Note that under these conditions G is unimodular, see [5, 1.4, 1.5, 1.9].)

Let $a \in G$. Then the automorphism $g \to aga^{-1}: G \to G$ induces an automorphism $\mathrm{ad}(a)$ of the Lie algebra $\mathfrak{G}$ of G. The latter splits into the direct sum of three $\mathrm{ad}(a)$ invariant subalgebras $\mathfrak{G}^s(a)$, $\mathfrak{G}^c(a)$, and $\mathfrak{G}^u(a)$ which correspond to the eigenvalues λ of $\mathrm{ad}(a)$ with $|\lambda| < 1$, $|\lambda| = 1$, and $|\lambda| > 1$ respectively. The element a is *semisimple* if $\mathrm{ad}(a)|\mathfrak{G}^c(a)$ is semisimple i.e. diagonalizable over the complex numbers. The subgroups of G corresponding to $\mathfrak{G}^s(a)$, $\mathfrak{G}^c(a)$, and $\mathfrak{G}^u(a)$ will be denoted $G^s(a)$, $G^c(a)$, and $G^u(a)$ respectively.

To apply the results of section 1 we let $X = \Gamma\backslash G = \{\Gamma g \mid g \in G\}$, $T = G^u$, and ϕ the map $\Gamma g \to \Gamma g a^{-1}: X \to X$ where a is a semisimple element of G. The fact that G is unimodular implies that the Haar measure on G induces a measure ω on X which is both T and ϕ invariant and which has the property that $\omega(N) > 0$ for all nonempty open subsets N of X. There remains the task of finding the set V such that the assumptions of 1.2 are satisfied.

2.1 Remarks. 1. The map $\Gamma g \to g^{-1}\Gamma: \Gamma\backslash G \to G/\Gamma = \{g\Gamma \mid g \in G\}$ is an isomorphism of the flow $(\Gamma\backslash G, G)$ onto the flow $(G, G/\Gamma)$. This map also provides an isomorphism of the flows (X, T) and (X, ϕ) onto $(G^u, G/\Gamma)$ and $(\psi, G/\Gamma)$ respectively where ψ is the map $g\Gamma \to ag\Gamma: G/\Gamma \to G/\Gamma$. Thus the results obtained there may be applied to the situation discussed in [2].

2. From the definition of G^u it follows that there is a neighborhood W of the identity of G^u such that the maps $g \to a^{-n}ga^n: W \to G^u$ converge uniformly to the constant map $g \to e: W \to G^u$. Thus any compact neighborhood V of e in $T = G^u$ with $V \subset W$ will satisfy assumption 1 of 1.2.

3. Let G^{sc} be the closed connected subgroup corresponding to $\mathfrak{G}^s(a) + \mathfrak{G}^c(a)$. The semisimplicity of a implies that given any neighborhood U of e in G^{sc} there exists a neighborhood W of e in G^{sc} with $a^nWa^{-n} \subset U$ $(n \geqq 0)$ (see [2]).

2.2 Lemma. *There exists an open neighborhood U of the identity of G such that the map $g \to xg: U \to X$ is a diffeomorphism onto an open neighborhood of x $(x \in X)$.*

Proof. It suffices to find a neighborhood U of e such that the map $g \to xg: U \to X$ is injective $(x \in X)$.

Since Γ is discrete there exists a neighborhood N of e with $\Gamma \cap N = e$. For each $g \in G$ let U_g be a symmetric neighborhood of e with $gU_g^4g^{-1} \subset N$. By the compactness of X there exists a finite subset F of G with $X = U\{\pi(gU_g) \mid g \in F\}$. (Here π is the canonical map of G onto $X = \Gamma\backslash G$.)

Set $U = \cap\{U_g \mid g \in F\}$ and let $xg_1 = xg_2$ for some $x \in X$ and g_1, g_2 in U. There exist $g \in F$ and $u \in U_g$ with $\pi(gu) = x$. Then $\Gamma gug_1 = \Gamma gug_2$ whence $gug_1g_2^{-1}u^{-1}g^{-1} \in \Gamma$. But $gug_1g_2^{-1}u^{-1}g^{-1} \in gU_gUUU_gg^{-1} \subset gU_g^4g^{-1}$ and so $gug_1g_2^{-1}u^{-1}g^{-1} = e$; from which it follows that $g_1 = g_2$. The proof is completed.

Let U be as in Lemma 2.2 and let W and Δ be neighborhoods of identity of G^u and G^{sc} respectively such that the maps $(t, \alpha) \to t\alpha: W \times \Delta \to G$ and $(\alpha, t) \to \alpha t: \Delta \times W \to G$ are injective with images contained in U (i.e. $W\Delta \cup \Delta W \subset U$). Then for any open subsets W_1 of W and Δ_1 of Δ the maps $(t, \alpha) \to xt\alpha: W_1 \times \Delta_1 \to X$ and $(\alpha, t) \to x\alpha t: \Delta_1 \times W_1 \to X$ are diffeomorphisms onto open subsets of x $(x \in X)$.

Henceforth V will denote a compact neighborhood of the identity of G^u and N an open neighborhood of the identity of G such that $t^{-1}gt \in W\Delta$ for all $g \in N$ and t in an open neighborhood of V. Then $gt = ts(g,t)\gamma(g,t)$ $(g \in N, t \in V)$ where s and γ are differentiable functions of $N \times V$ into W and Δ respectively.

Let ψ be the function $(g,t) \to ts(g,t): N \times V \to T$, ψ_g the map $t \to \psi(g,t): V \to T$ and ν_g the measure $\mu\psi_g^{-1}$ $(g \in N)$. Then

$$\frac{1}{\mu(V)} \int_V f(xts(g,t)\phi^n)d\mu(t) = \frac{1}{\mu(V)} \int_{\psi_g(V)} f(xt\phi^n)d\nu_g(t)$$

$$= \frac{1}{\mu(V)} \int_{\psi_g(V)} f(xt\phi^n) \frac{d\nu_g}{d\mu}\, d\mu(t).$$

Thus

$$
(1)\qquad \begin{aligned}
&\left| \frac{1}{\mu(V)} \int_V f(xts(g,t)\phi^n)d\mu(t) - \frac{1}{\mu(V)} \int_V f(xt\phi^n)d\mu(t) \right| \\
&\qquad \leqq \frac{1}{\mu(V)} \left(\int_V \left| 1 - \frac{d\nu_g}{d\mu} \right| |f| \, d\mu + \int_{V\Delta\psi_g(V)} \left| 1 + \frac{d\nu_g}{d\mu} \right| |f| \, d\mu \right).
\end{aligned}
$$

Since ψ is continuously differentiable and ψ_e is the inclusion map, $\lim_{g\to e} \mu(V\Delta\psi_g(V)) = 0$ and $\lim_{g\to e} |1 - d\nu_g/d\mu| = 0$ uniformly.

Let $\varepsilon > 0$. There exists a neighborhood M of the identity of G such that (i) $|f(xg) - f(x)| < \varepsilon/2$ $(x \in X, g \in M)$. Since $\gamma(e,t) = e$ $(t \in V)$ there exists a neighborhood N_1 of e in G such that $N_1 \subset N$ and (ii) $\phi^{-n}\gamma(g,t)\phi^n \in M$ $(g \in N_1$ $t \in V, n \geqq 0)$ (see 3 of 2.1). Let K be a neighborhood of e in G such that $K \subset N_1$ and (iii) $|1 - d\nu_g/d\mu| |f| < \varepsilon/4$ and $\mu(V\Delta\psi_g(V)) \| f | < \varepsilon/4$ $(g \in K)$.

Let $x \in X$ and $y = xg$ for some $g \in K$. Then

$$
f(yt\phi^n) = f(xgt\phi^n) = f(xts(g,t)\gamma(g,t)\phi^n) = f(xts(g,t)\phi^n\phi^{-n}\gamma(g,t)\phi^n).
$$

Hence (a) $|f(yt\phi^n - f(xts(g,t)\phi^n)| < \varepsilon/2$ $(t \in V, n \geqq 0)$ by (i) and (ii). Combining (a), (1) and (iii) we see that

$$
\left| \frac{1}{\mu(V)} \int_V f(xt\phi^n)d\mu(t) - \frac{1}{\mu(V)} \int_V f(yt\phi^n)d\mu(t) \right| < \varepsilon \qquad (n \geqq 0).
$$

We have thus proved:

2.3 Lemma. *There exists a compact neighborhood V of the identity of T such that the sequence $(LU^n f \mid n \geqq 0)$ is equicontinuous $(f \in C(X))$ where $Lh(x) = (1/\mu(V))\int_V h(xt)d\mu(t)$ and $Uh(x) = h(x\phi)(x \in X, h \in C(X))$.*

Lemma 2.3 and 2 of 2.1 allow us to apply 1.10 and conclude:

2.4 Proposition. *Let a be a semisimple element of a unimodular connected Lie group G, Γ a cocompact discrete subgroup of G, T_a the map $g\Gamma \to ag\Gamma$: $G/\Gamma \to G/\Gamma$, and ω the measure induced on G/Γ by the Haar measure on G. Then the following statements are pairwise equivalent.* (1) *$(T_a, G/\Gamma, \omega)$ is weak-mixing.* (2) *$(T_a, G/\Gamma)$ is topologically weak-mixing.* (3) *$(G^u, G/\Gamma)$ is uniquely ergodic.* (4) *$(G^u, G/\Gamma, \omega)$ is ergodic.* (5) *$(G^u, G/\Gamma)$ is point transitve.* (6) *$(G^u, G/\Gamma)$ is minimal.* (For (5) recall that G^u is nilpotent [2] hence amenable.)

3. Semisimple G

In this section we add the assumption that G is semisimple with finite center and no compact factors. (This means that G modulo its center is isomorphic to a product of simple groups none of which is compact.) We conclude that the statement "$(T_a, G/\Gamma, \omega)$ is ergodic" may be added to those of 2.4.

We preserve the notation of section 2 and in addition denote by H the closed subgroup of G generated by $G^u \cup G^s$.

3.1 LEMMA. *Let f be a continuous ϕ-eigenfunction on X. Then $f(xh) = f(x)$ $(x \in X, h \in H)$.*

PROOF. Let $f(x\phi) = \lambda f(x)$ $(x \in X)$. Then $\lambda^n f(xg) = f(xg\phi^n) = f(x\phi^n\phi^{-n}g\phi^n)$ $(x \in X, g \in G$, all $n)$.

If $g \in G^u$ then one may choose a sequence $n_i \to -\infty$ with $\lambda^{n_i} \to 1$. Since $\phi^{-n_i}g\phi^{n_i} \to e$ the above relation shows that $f(xg) = f(x)$ $(x \in X)$.

On the other hand if $g \in G^s$ one takes a sequence $n_i \to \infty$ with $\lambda^{n_i} \to 1$ to show that in this case also $f(xg) = g(x)$ $(x \in X)$.

The proof is completed.

3.2 LEMMA. *Let K be the closed subgroup of G generated by $\Gamma \cup H$. Then $(K\backslash G, \phi)$ is equicontinuous.*

PROOF. Let d be a metric on $K\backslash G$ and $\varepsilon > 0$. Then there exists a neighborhood U of e in G such that $d(xg, x) < \varepsilon$ $(x \in K\backslash G, g \in U)$. By 3 of 2.1 there exists a neighborhood W of e in G^{sc} with $\phi^{-n}g\phi^n \in U$ $(n \geqq 0, g \in W)$.

Let $x \in K\backslash G$ and $y = xg$ for some $g \in W$. Then

$$d(x\phi^n, y\phi^n) = d(x\phi^x, x\phi^n\phi^{-n}g\phi^n) < \varepsilon \qquad (n \geqq 0).$$

Since xW is a neighborhood of x, the sequence $(\phi^n \mid n \geqq 0)$ is equicontinuous on $K\backslash G$. The proof is completed.

3.3 REMARKS. 1. The subgroup H is normal and so $K = \overline{\Gamma H}$. (For a proof see [6]. There the case of a real flow is considered, the proof for the discrete case is similar.)

2. Set $S = G/H$ and let $\Delta = \overline{\pi(\Gamma)}$ where $\pi: G \to S$ is the canonical map. Then S is semisimple and S/Δ is compact. Hence by [1] the component Δ_0 of the identity of Δ is a normal subgroup of S. Set $R = S/\Delta_0$ and $\Sigma = \Delta/\Delta_0$. Then Σ is a closed discrete subgroup of the semisimple group R and $\Sigma\backslash R$ is compact; indeed $\Sigma\backslash R \cong \Delta\backslash S$. Thus ϕ and ω induce a homeomorphism ψ of $\Sigma\backslash R$ and a measure of ν on $\Sigma\backslash R$ such that $(\Sigma\backslash R, \psi, \omega)$ is a homomorphic image of $(\Gamma\backslash G, \phi, \omega)$.

Since $\pi(K) \subset \Delta$, $(\Sigma\backslash R, \psi)$ is a homomorphic image of $(K\backslash G, \phi)$ whence by 3.2 it is equicontinuous.

If $(\Gamma\backslash G, \phi, \omega)$ were ergodic then $(\Sigma\backslash R, \psi, \omega)$ would be ergodic and equicontinuous hence minimal (recall $\omega(N) > 0$ for all nonempty open sets). This would mean that $\Sigma\backslash R$ was homeomorphic to an n-torus or a single point.

However $\Sigma\backslash R$ cannot be homeomorphic to an n-torus with $n \geqq 1$, for then Σ would be abelian (consider the homotopy sequence of the bundle $(R, \Sigma\backslash R)$ and recall that R is connected) and so would be in the center of R [1, 4.4] and $\Sigma\backslash R$ would be a torus; contradicting the fact that R is semisimple.

Thus if $(\Gamma\backslash G, \phi, \omega)$ is ergodic, then $\Sigma = R$ whence $\Delta = S$ and so $K = G$.

3.4 PROPOSITION. *Let a be a semisimple element of a connected semisimple Lie group G with finite center and no compact factors, Γ a cocompact discrete subgroup of G, T_a the map $g\Gamma \to ag\Gamma$: $G/\Gamma \to G/\Gamma$, and ω the measure induced on G/Γ by the Haar measure on G. Then the following statements are pairwise equivalent.* (1) *$(T_a, G/\Gamma, \omega)$ is weak-mixing.* (2) *$(T_a, G/\Gamma)$ is topologically weak-mixing.* (3) *$(G^u, G/\Gamma)$ is uniquely ergodic.* (4) *$(G^u, G/\Gamma, \omega)$ is ergodic.* (5) *$(G^u, G/\Gamma)$ is point transitive.* (6) *$(G^u, G/\Gamma)$ is minimal.* (7) *$(T_a, G/\Gamma, \omega)$ is ergodic.*

PROOF. The equivalence of the first six statements follows from 2.4.

Clearly (7) follows from (1). We show that (7) implies (2).

Let f be a continuous T_a-eigenfunction. Then by 3.1 $f(hg\Gamma) = f(g\Gamma)$ $(g \in G, h \in H)$ whence f is constant on G/Γ by 2 of 3.2. The proof is completed.

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